Manipulation of Attractors as a Recurrent Network Paradigm

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Ising Model

• Spin is a microscopic magnetic moment in an atom.

• Ferromagnetism of Iron was explained with the Ising model.

• $H(S) = -\sum J S_i S_j - h \sum S_i$, $S_i$ is a spin and $h$ is an external magnetic field.

• $J > 0$ in ferromagnet and $J < 0$ in antiferromagnet
Spin Glass

- \( H = - \sum_{<ij>} J_{ij} S_i S_j - h \sum_i S_i \)
- \( J_{ij} \) has a value which depends on \( i \) and \( j \).
- There exist many minima.
Additive Model of a Neuron

\[ C_j \frac{dv_j(t)}{dt} + \frac{v_j(t)}{R_j} = \sum_{i=1}^{N} w_{ji} x_i(t) + I_j \]  \quad (1)

\[ x_j(t) = \varphi(v_j(t)) \]
Architectural Graph of a Hopfield Network

Recurrent network allows a feedback loop.
The Energy (Lyapunov) function (1/4)

- \( E = -\frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} w_{ji} x_i x_j + \sum_{j=1}^{N} \frac{1}{R_j} \int_{0}^{x_j} \varphi_j^{-1}(x) dx - \sum_{j=1}^{N} I_j x_j \)

- Differentiating \( E \) with respect to time, we get

\[
\frac{dE}{dt} = -\sum_{j=1}^{N} \left( \sum_{i=1}^{N} w_{ji} x_i - \frac{v_j}{R_j} + I_j \right) \frac{dx_j}{dt}
\]

\[
\frac{dE}{dt} = -\sum_{j=1}^{N} C_j \left( \frac{dv_j}{dt} \right) \frac{dx_j}{dt}
\]

\[
\frac{dE}{dt} = -\sum_{j=1}^{N} C_j \left[ \frac{d}{dt} \varphi_j^{-1}(x_j) \right] \frac{dx_j}{dt}
\]

\[
= -\sum_{j=1}^{N} C_j \left( \frac{dx_j}{dt} \right)^2 \left[ \frac{d}{dx_j} \varphi_j^{-1}(x_j) \right]
\]
The Energy (Lyapunov) function (2/4)

1. The energy function $E$ is a Lyapunov function of the continuous Hopfield model.
2. The model is stable in accordance with Lyapunov’s Theorem 1.
   
   **Theorem 1.** The equilibrium state $\bar{x}$ is stable if in a small neighborhood of $\bar{x}$ there exists a positive definite function $V(x)$ such that its derivative with respect to time is semidefinite in that region.
The Energy (Lyapunov) function (3/4)

- $V(x)$ is positive definite in the state space $L$ if, for all $x$ in $L$, it satisfies the following requirements:
  - 1. $V(x)$ has continuous partial derivatives with respect to the elements of the state vector $x$.
  - 2. $V(x) = 0$.
  - 3. $V(x) > 0$ if $x \neq \bar{x}$

- According to Theorem 1, if $V(x)$ is a Lyapunov function, the equilibrium state $\bar{x}$ is stable if \( \frac{d}{dt}V(x) < 0 \) for $x \in U - \bar{x}$
The Energy (Lyapunov) function (4/4)

- The time evolution of the continuous Hopfield model described by the nonlinear first-order differential equation (1) represents a trajectory in state space, which seeks out the minima of the energy (Lyapunov) function $E$ and comes to a stop at such fixed points.
The Discrete Hopfield (1/5)

• Content-Addressable Memory
• Storage Phase. We wish to store a set of \( N \)-dimensional vectors, denoted by \( \{\xi_\mu | \mu = 1, 2, ..., M\} \).
• Synaptic weight from neuron \( i \) to neuron \( j \) is defined by

\[
 w_{ji} = \frac{1}{N} \sum_{\mu=1}^{M} \xi_{\mu,j} \xi_{\mu,i}
\]
The Discrete Hopfield (2/5)

• We set $w_{ii} = 0$ for all $i$, which means that the neuron has no self-feedback.

• Then the synaptic weight matrix is

$$ W = \frac{1}{N} \sum_{\mu=1}^{M} \xi_{\mu} \xi_{\mu}^T - MI. $$

• Retrieval Phase. The state updating from one iteration to the next is deterministic, but the selection of a neuron to perform the updating is done randomly, i.e., asynchronous learning.
The Discrete Hopfield (3/5)

• \( y(t + 1) = \text{sgn}(Wy(t) + b) \), where \( \text{sgn}(x) \) is a signum function.

• \( \text{sgn}(x) = x/|x| \), if \( x \neq 0 \) and \( \text{sgn}(x) = 0 \), if \( x = 0 \).

• Example.

\[
\mathbf{w} = \frac{1}{3} \begin{bmatrix} +1 \\ -1 \\ +1 \end{bmatrix} [+1,-1,1] + \frac{1}{3} \begin{bmatrix} -1 \\ +1 \\ -1 \end{bmatrix} [-1,1,-1] - \frac{2}{3} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & -2 & +2 \\ -2 & 0 & -2 \\ +2 & -2 & 0 \end{bmatrix}.
\]
The Discrete Hopfield (4/5)
The Discrete Hopfield (5/5)

\[ w_y = \frac{1}{3} \begin{bmatrix} 0 & -2 & +2 \\ -2 & 0 & -2 \\ +2 & -2 & 0 \end{bmatrix} \begin{bmatrix} +1 \\ -1 \\ +1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} +4 \\ -4 \\ +4 \end{bmatrix}, \text{ limiting this result yields} \]

\[ \text{sgn}[w_y] = \begin{bmatrix} +1 \\ -1 \\ +1 \end{bmatrix} = y. \]
Computer Experiment (1/2)

• Set of handcrafted patterns

Correct recollection of corrupted 3  Incorrect recollection of corrupted 2
Computer Experiment (2/2)

- Load parameter $\alpha = \frac{M}{N}$
- Critical value $\alpha_c = 0.14$
- Mixture states: a state formed out of three fundamental memories. Row 6, column 4: negative of digit 1, digit 4, and digit 9.
- Spin-glass states: local minima of the energy landscape that are not correlated with any of the fundamental memories of the network. Row 7, column 6.
Reference