Section Summary

- Definition of a Matrix
- Matrix Arithmetic
- Transposes and Powers of Arithmetic
- Zero-One matrices
Matrices

Matrices are useful discrete structures that can be used in many ways. For example, they are used to:

- describe certain types of functions known as linear transformations.
- Express which vertices of a graph are connected by edges (see Chapter 10).
- In later chapters, we will see matrices used to build models of:
  - Transportation systems.
  - Communication networks.
- Algorithms based on matrix models will be presented in later chapters.
- Here we cover the aspect of matrix arithmetic that will be needed later.
Matrix

**Definition:** A *matrix* is a rectangular array of numbers. A matrix with *m* rows and *n* columns is called an *m*×*n* matrix.

- The plural of matrix is *matrices*.
- A matrix with the same number of rows as columns is called *square*.
- Two matrices are *equal* if they have the same number of rows and the same number of columns and the corresponding entries in every position are equal.

\[
\begin{bmatrix}
1 & 1 \\
0 & 2 \\
1 & 3
\end{bmatrix}
\]

3×2 matrix
Notation

- Let \( m \) and \( n \) be positive integers and let

\[
A = \begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn}
\end{bmatrix}
\]

- The \( i \)th row of \( A \) is the \( 1 \times n \) matrix \([a_{i1}, a_{i2}, \ldots, a_{in}]\). The \( j \)th column of \( A \) is the \( m \times 1 \) matrix:

\[
\begin{bmatrix}
a_{1j} \\
a_{2j} \\
\vdots \\
a_{mj}
\end{bmatrix}
\]

- The \((i,j)\)th element or entry of \( A \) is the element \( a_{ij} \). We can use \( A = [a_{ij}] \) to denote the matrix with its \((i,j)\)th element equal to \( a_{ij} \).
Matrix Arithmetic: Addition

**Definition:** Let \( A = [a_{ij}] \) and \( B = [b_{ij}] \) be \( m \times n \) matrices. The sum of \( A \) and \( B \), denoted by \( A + B \), is the \( m \times n \) matrix that has \( a_{ij} + b_{ij} \) as its \((i,j)\)th element. In other words, \( A + B = [a_{ij} + b_{ij}] \).

**Example:**

\[
\begin{bmatrix}
1 & 0 & -1 \\
2 & 2 & -3 \\
3 & 4 & 0
\end{bmatrix}
+ \begin{bmatrix}
3 & 4 & -1 \\
1 & -3 & 0 \\
-1 & 1 & 2
\end{bmatrix}
= \begin{bmatrix}
4 & 4 & -2 \\
3 & -1 & -3 \\
2 & 5 & 2
\end{bmatrix}
\]

Note that matrices of different sizes can not be added.
Matrix Multiplication

**Definition:** Let $A$ be an $n \times k$ matrix and $B$ be a $k \times n$ matrix. The product of $A$ and $B$, denoted by $AB$, is the $m \times n$ matrix that has its $(i,j)$th element equal to the sum of the products of the corresponding elements from the $i$th row of $A$ and the $j$th column of $B$. In other words, if $AB = [c_{ij}]$ then $c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{kj}b_{2j}$.

**Example:**

$$\begin{bmatrix}
1 & 0 & 4 \\
2 & 1 & 1 \\
3 & 1 & 0 \\
0 & 2 & 2
\end{bmatrix} \begin{bmatrix}
2 & 4 \\
1 & 1 \\
3 & 0
\end{bmatrix} = \begin{bmatrix}
14 & 4 \\
8 & 9 \\
7 & 13 \\
8 & 2
\end{bmatrix}$$

The product of two matrices is undefined when the number of columns in the first matrix is not the same as the number of rows in the second.
Illustration of Matrix Multiplication

- The Product of $A = [a_{ij}]$ and $B = [b_{ij}]$

$$A = \begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1k} \\
a_{21} & a_{22} & \cdots & a_{2k} \\
\vdots & \vdots & \ddots & \vdots \\
a_{i1} & a_{i2} & \cdots & a_{ik} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mk}
\end{bmatrix} \quad B = \begin{bmatrix}
b_{11} & a_{12} & \cdots & b_{1j} & \cdots & b_{1n} \\
b_{21} & b_{22} & \cdots & b_{2j} & \cdots & b_{2n} \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
b_{k1} & b_{k2} & \cdots & b_{kj} & \cdots & b_{kn}
\end{bmatrix}$$

$$AB = \begin{bmatrix}
c_{11} & c_{12} & \cdots & c_{1n} \\
c_{21} & c_{22} & \cdots & c_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
c_{i1} & c_{i2} & \cdots & c_{ij} \\
\vdots & \vdots & \ddots & \vdots \\
c_{m1} & c_{m2} & \cdots & c_{mn}
\end{bmatrix}$$

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ik}b_{kj}$$
Matrix Multiplication is not Commutative

Example: Let

\[ A = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \]

Does \( AB = BA \)?

Solution:

\[ AB = \begin{bmatrix} 2 & 2 \\ 5 & 3 \end{bmatrix} \quad BA = \begin{bmatrix} 4 & 3 \\ 3 & 2 \end{bmatrix} \]

\( AB \neq BA \)
Identity Matrix and Powers of Matrices

**Definition**: The *identity matrix of order n* is the $m \times n$ matrix $I_n = [\delta_{ij}]$, where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$.

$$I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

$$AI_n = I_mA = A$$

when $A$ is an $m \times n$ matrix

Powers of square matrices can be defined. When $A$ is an $n \times n$ matrix, we have:

$$A^0 = I_n \quad A^r = \underbrace{AAA\cdots A}_{r \text{ times}}$$
Transposes of Matrices

**Definition:** Let $A = [a_{ij}]$ be an $m \times n$ matrix. The *transpose* of $A$, denoted by $A^t$, is the $n \times m$ matrix obtained by interchanging the rows and columns of $A$.

If $A^t = [b_{ij}]$, then $b_{ij} = a_{ji}$ for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$.

The transpose of the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ is the matrix $\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$. 
Transposes of Matrices

**Definition:** A square matrix $A$ is called symmetric if $A = A^t$. Thus $A = [a_{ij}]$ is symmetric if $a_{ij} = a_{ji}$ for $i$ and $j$ with $1 \leq i \leq n$ and $1 \leq j \leq n$.

The matrix $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ is square.

Square matrices do not change when their rows and columns are interchanged.
Zero-One Matrices

**Definition:** A matrix all of whose entries are either 0 or 1 is called a *zero-one matrix*. (These will be used in Chapters 9 and 10.)

Algorithms operating on discrete structures represented by zero-one matrices are based on Boolean arithmetic defined by the following Boolean operations:

\[
b_1 \land b_2 = \begin{cases} 1 & \text{if } b_1 = b_2 = 1 \\ 0 & \text{otherwise} \end{cases} \quad b_1 \lor b_2 = \begin{cases} 1 & \text{if } b_1 = 1 \text{ or } b_2 = 1 \\ 0 & \text{otherwise} \end{cases}
\]
Zero-One Matrices

**Definition:** Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be an $m \times n$ zero-one matrices.

- The *join* of $A$ and $B$ is the zero-one matrix with $(i,j)$th entry $a_{ij} \lor b_{ij}$. The *join* of $A$ and $B$ is denoted by $A \lor B$.
- The *meet* of $A$ and $B$ is the zero-one matrix with $(i,j)$th entry $a_{ij} \land b_{ij}$. The *meet* of $A$ and $B$ is denoted by $A \land B$. 
Joins and Meets of Zero-One Matrices

**Example:** Find the join and meet of the zero-one matrices

\[
A = \begin{bmatrix}
1 & 0 & 1 \\
0 & 1 & 0 \\
\end{bmatrix}, \quad B = \begin{bmatrix}
0 & 1 & 0 \\
1 & 1 & 0 \\
\end{bmatrix}.
\]

**Solution:** The join of \( A \) and \( B \) is

\[
A \lor B = \begin{bmatrix}
1 \lor 0 & 0 \lor 1 & 1 \lor 0 \\
0 \lor 1 & 1 \lor 1 & 0 \lor 0 \\
\end{bmatrix} = \begin{bmatrix}
1 & 1 & 1 \\
1 & 1 & 0 \\
\end{bmatrix}.
\]

The meet of \( A \) and \( B \) is

\[
A \land B = \begin{bmatrix}
1 \land 0 & 0 \land 1 & 1 \land 0 \\
0 \land 1 & 1 \land 1 & 0 \land 0 \\
\end{bmatrix} = \begin{bmatrix}
0 & 0 & 0 \\
0 & 1 & 0 \\
\end{bmatrix}.
\]
Boolean Product of Zero-One Matrices

**Definition:** Let $A = [a_{ij}]$ be an $m \times k$ zero-one matrix and $B = [b_{ij}]$ be a $k \times n$ zero-one matrix. The **Boolean product** of $A$ and $B$, denoted by $A \odot B$, is the $m \times n$ zero-one matrix with $(i,j)$th entry

$$c_{ij} = (a_{i1} \land b_{1j}) \lor (a_{i2} \land b_{2j}) \lor \ldots \lor (a_{ik} \land b_{kj}).$$

**Example:** Find the Boolean product of $A$ and $B$, where

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \quad B = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$
Boolean Product of Zero-One Matrices

**Solution:** The Boolean product $A \odot B$ is given by

$$A \odot B = \begin{bmatrix}
(1 \land 1) \lor (0 \land 0) & (1 \land 1) \lor (0 \land 1) & (1 \land 0) \lor (0 \land 1) \\
(0 \land 1) \lor (1 \land 0) & (0 \land 1) \lor (1 \land 1) & (0 \land 0) \lor (1 \land 1) \\
(1 \land 1) \lor (0 \land 0) & (1 \land 1) \lor (0 \land 1) & (1 \land 0) \lor (0 \land 1)
\end{bmatrix}$$

$$= \begin{bmatrix}
1 \lor 0 & 1 \lor 0 & 0 \lor 0 \\
0 \lor 0 & 0 \lor 1 & 0 \lor 1 \\
1 \lor 0 & 1 \lor 0 & 0 \lor 0
\end{bmatrix}$$

$$= \begin{bmatrix}
1 & 1 & 0 \\
0 & 1 & 1 \\
1 & 1 & 0
\end{bmatrix}.$$
Boolean Powers of Zero-One Matrices

**Definition:** Let $A$ be a square zero-one matrix and let $r$ be a positive integer. The $r$th Boolean power of $A$ is the Boolean product of $r$ factors of $A$, denoted by $A^{[r]}$. Hence, \[ A^{[r]} = \underbrace{A \circ A \circ \ldots \circ A}_{r \text{ times}}. \]

We define $A^{[r]}$ to be $I_n$.

(The Boolean product is well defined because the Boolean product of matrices is associative.)
Boolean Powers of Zero-One Matrices

Example: Let \( A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} \).

Find \( A^n \) for all positive integers \( n \).

Solution:

\[ A^{[2]} = A \odot A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \quad A^{[3]} = A^{[2]} \odot A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \]

\[ A^{[4]} = A^{[3]} \odot A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \]

\[ A^{[5]} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \quad A^{[n]} = A^5 \quad \text{for all positive integers } n \quad \text{with } n \geq 5. \]