Course 4190.101.001
Discrete Mathematics
Homework 2: Sets, Functions, Sequences, Sums, and Matrices

April 4, 2017

Due: April 13, 2017, 23:59

Most of the questions here are selected from the textbook and these questions should only be used for your homework assignment. Your documents should be submitted as a **single** `pdf` file named “`hw2_STUDENT-ID.pdf`”. For example, `hw2_2017-00000.pdf`. Email your `pdf` file to `snu.dm001@gmail.com` with title as “`HW2, STUDENT-ID, name`”.

1. Translate each of these quantifications into English and determine its truth value.

   a) $\forall x \in \mathbb{R}\ (x^2 \neq -1)$

   b) $\exists x \in \mathbb{Z}\ (x^2 = 2)$

   c) $\forall x \in \mathbb{Z}\ (x^2 > 0)$

   d) $\exists x \in \mathbb{R}\ (x^2 = x)$

2. Find the truth set of each of these predicates where the domain is the set of integers.

   a) $P(x)$: “$x^3 \geq 1$”

   b) $Q(x)$: “$x^2 = 2$”

   c) $R(x)$: “$x < x^2$”

3. The **symmetric difference** of A and B, denoted by $A \oplus B$, is the set containing those elements in either A or B, but not in both A and B. Show that $A \oplus B = (A - B) \cup (B - A)$.

4. Determine whether each of these sets is the power set of a set, where a and b are distinct elements.

   a) ϕ

   b) $\{\phi, \{a\}\}$

   c) $\{\phi, \{a\}, \{\phi, a\}\}$

   d) $\{\phi, \{a\}, \{b\}, \{a, b\}\}$

5. Let A and B be the multisets $\{3 \cdot a, 2 \cdot b, 1 \cdot c\}$ and $\{2 \cdot a, 3 \cdot b, 4 \cdot c\}$, respectively. Find

   a) $A \cup B$.
   b) $A \cap B$.
   c) $A - B$.
   d) $B - A$.

6. Let $f(x) = ax + b$ and $g(x) = cx + d$, where a, b, c, and d are constants. Determine for which constants a, b, c, and d it is true that $f \circ g = g \circ f$.

7. If f and $f \circ g$ are one-to-one, does it follow that g is one-to-one? Justify your answer.

8. If f and $f \circ g$ are onto, does it follow that g is onto? Justify your answer.

9. Draw graphs of each of these functions.

   a) $f(x) = \lfloor 2x + 1 \rfloor$
   b) $f(x) = \lceil 1/x \rceil$
   c) $f(x) = \lfloor 2x \rfloor \lceil x/2 \rceil$
   d) $f(x) = \lceil \lfloor x - \frac{1}{2} \rfloor + \frac{1}{2} \rceil$

10. Prove or disprove each of these statements about the floor and ceiling functions.

    a) $\lceil \lfloor x \rfloor \rceil = \lfloor x \rfloor$ for all real numbers x.
    b) $\lfloor 2x \rfloor = 2 \lfloor x \rfloor$ whenever x is a real number.
    c) $\lceil x \rceil + \lceil y \rceil - \lceil x + y \rceil = 0$ or 1 whenever x and y are real numbers.

11. Suppose that f is a function from A to B, where A and B are finite sets with $|A| = |B|$. Show that f is one-to-one if and only if it is onto.

12. Use the identity $k^2 - (k-1)^2 = 2k - 1$ and the equation $\sum_{j=1}^{n} (a_j - a_{j-1}) = a_n - a_0$ to find

    a) a formula for $\sum_{k=1}^{n} (2k - 1)$.
    b) a formula for $\sum_{k=1}^{n} k$.

13. Determine whether each of these sets is countable or uncountable. For those that are countable, exhibit a one-to-one correspondence between the set of natural numbers and that set.

    a) integers not divisible by 3
    b) integers divisible by 5 but not by 7
    c) the real numbers with decimal representations consisting of all 1s
    d) the real numbers with decimal representations of all 1s or 9s

14. Show that if A and B are sets, A is uncountable, and $A \subseteq B$, then B is uncountable.

15. Show that the set of all finite bit strings is countable.

16. Let $\mathbf{A}$ be an invertible matrix. Show that $(\mathbf{A}^n)^{-1} = (\mathbf{A}^{-1})^n$ whenever n is a positive integer.

17. Let $\mathbf{A}$ be a matrix. Show that the matrix $\mathbf{A}\mathbf{A}^\mathsf{T}$ is symmetric. [*Hint*: $(\mathbf{AB})^\mathsf{T} = \mathbf{B}^\mathsf{T}\mathbf{A}^\mathsf{T}$]