Course 4190.101.001
Discrete Mathematics
Homework 4: Induction and Recursion

May 2, 2017

Due: May 11, 2017, 23:59

Most of the questions here are selected from the textbook and these questions should only be used for your homework assignment. Your documents should be submitted as a **single** `pdf` file named "`hw4_STUDENT-ID.pdf`". For example, `hw4_2017-00000.pdf`. Email your `pdf` file to `snu.dm001@gmail.com` with title as "`HW4, STUDENT-ID, name`".

1. Prove that $1/(2n) \leq [1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)]/(2 \cdot 4 \cdot \cdots \cdot 2n)$ whenever n is a positive integer.

2. What is wrong with this "proof"?
   "Theorem" For every positive integer n, if x and y are positive integers with $\max(x, y) = n$, then $x = y$.
   Basis Step: Suppose that $n = 1$. If $\max(x, y) = 1$ and y are positive integers, we have $x = 1$ and $y = 1$.
   Inductive Step: Let k be a positive integer. Assume that whenever $\max(x, y) = k$ and x and y are positive integers, then $x = y$. Now let $\max(x, y) = k + 1$, where x and y are positive integers. Then $\max(x - 1, y - 1) = k$, so by the inductive hypothesis, $x - 1 = y - 1$. It follows that $x = y$, completing the inductive step.

3. Consider this variation of the game of Nim. The game begins with n matches. Two players take turns removing matches, one, two, or three at a time. The player removing the last match loses. Use strong induction to show that if each player plays the best strategy possible, the first player wins if $n = 4j, 4j + 2$,or $4j + 3$ for some nonnegative integer j and the second player wins in the remaining case when $n = 4j + 1$ for some nonnegative integer j.

4. Using the well-ordering principle to show that if x and y are real numbers with $x < y$, then there is a rational number r with $x < r < y$. [*Hint:* Use the Archimedean property, given in Appendix 1, to find a positive integer A with $A > 1/(y - x)$. Then show that

there is a rational number r with denominator A between x and y by looking at the numbers $\lfloor x \rfloor + j/A$, where j is a positive integer.]

5. Let S be the subset of the set of ordered pairs of integers defined recursively by
   Basis step: $(0, 0) \in S$.
   Recursive step: If $(a, b) \in S$, then $(a + 2, b + 3) \in S$ and $(a + 3, b + 2) \in S$.

   a) List the elements of S produced by the first five applications of the recursive definition.

   b) Use strong induction on the number of applications of the recursive step of the definition to show that $5|a + b$ ("$a + b$" can be divided by 5) when $(a, b) \in S$.

   c) Use structural induction to show that $5|a + b$ when $(a, b) \in S$.

6. Use generalized induction to show that if $a_{m,n}$ is defined recursively by $a_{1,1} = 5$ and

$$a_{m,n} = \begin{cases} a_{m-1,n} + 2 & \text{if } n = 1 \text{ and } m > 1 \\ a_{m,n-1} + 2 & \text{if } n > 1, \end{cases} \tag{1}$$

   then $a_{m,n} = 2(m + n) + 1$ for all $(m, n) \in \mathbb{Z}^+ \times \mathbb{Z}^+$.

7. Devise a recursive algorithm to find the nth term of the sequence defined by $a_0 = 1$, $a_1 = 2$, and $a_n = a_{n-1} \cdot a_{n-2}$, for $n = 2, 3, 4, ...$

8. Use a merge sort to sort $4, 8, 2, 7, 1, 6, 3, 5$. Show all the steps used by the algorithm.