Discrete Mathematics

HW #1 Solution

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1. Let p and q be the propositions “Swimming at the New Jersey shore is allowed” and “Sharks have been spotted near the shore,” respectively. Express each of these compound propositions as an English sentence.

   p : swimming at the New Jersey shore is allowed
   q : sharks have been spotted near the shore

   a) $p \land q$

   Swimming at the New Jersey shore is allowed and sharks have been spotted near the shore.

   b) $p \rightarrow \neg q$

   If swimming at the New Jersey shore is allowed, then sharks haven't been spotted near the shore.

1. Let p and q be the propositions "Swimming at the New Jersey shore is allowed" and "Sharks have been spotted near the shore," respectively. Express each of these compound propositions as an English sentence.

p : swimming at the New Jersey shore is allowed
q : sharks have been spotted near the shore

c) $\neg q \rightarrow p$

If sharks haven't been spotted near the shore,
then swimming at the New Jersey shore is allowed.

d) $\neg p \rightarrow \neg q$

If swimming at the New Jersey shore isn't allowed,
then sharks haven't been spotted near the shore.

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   p : swimming at the New Jersey shore is allowed
   q : sharks have been spotted near the shore

   e) $p \leftrightarrow \neg q$

   Swimming at the New Jersey is allowed if and only if sharks haven't been spotted near the shore.

   f) $\neg p \wedge (p \vee \neg q)$

   Swimming at the New Jersey shore isn't allowed, and either swimming at the New Jersey shore is allowed or sharks haven't been spotted near the shore.

1. Let p and q be the propositions "Swimming at the New Jersey shore is allowed" and "Sharks have been spotted near the shore," respectively. Express each of these compound propositions as an English sentence.

p : swimming at the New Jersey shore is allowed
q : sharks have been spotted near the shore

e) $p \leftrightarrow \neg q$

Swimming at the New Jersey is allowed if and only if sharks haven't been spotted near the shore.

f) $\neg p \land (p \lor \neg q) \equiv (\neg p \land p) \lor (\neg p \land \neg q) \equiv (\neg p \land \neg q)$

Swimming at the New Jersey shore isn't allowed, and sharks haven't been spotted near the shore.

2. Let p and q be the propositions

p : It is below freezing.

q : It is snowing.

Write these propositions using p and q and logical connectives (including negations).

a) It is below freezing but not snowing.

$p \land \neg q$

b) If it is below freezing, it is also snowing.

$p \rightarrow q$

c) Either it is below freezing or it is snowing, but it is not snowing if it is below freezing.

$(p \lor q) \land (p \rightarrow \neg q)$

d) That it is below freezing is necessary and sufficient for it to be snowing.

$p \leftrightarrow q$

3. State the converse, contrapositive, and inverse of each of these conditional statements.

a) If it snows tonight, then I will stay at home.

converse : If I stay at home, then it will snow tonight.
contrapositive : If I don’t stay at home, then it won’t snow tonight.
inverse : If it doesn't snow tonight, then I won't stay at home.

b) I go to the beach whenever it is a sunny summer day.

converse : Whenever I go to the beach, it is a sunny summer day.
contrapositive : Whenever I don't go to the beach, it isn't a sunny summer day.
inverse : Whenever it isn't a sunny summer day, I don't go to the beach.

c) When I stay up late, it is necessary that I sleep until noon.

converse : If I sleep until noon, then I stayed up late.
contrapositive : If I don’t sleep until noon, then I didn't stay up late.
inverse : If I don't stay up late, then I don’t sleep until noon.

4. What is the value of x after each of these statements is encountered in a computer program, if $x = 1$ before the statement is reached?

a) **if** $1 + 1 = 2$ **then** $x \coloneqq x + 1$

$True$ $\quad x \; becomes \; x + 1 \; = \; 2$

b) **if** $(2 + 3 = 5) \; AND \; (3 + 4 = 7)$ **then** $x \coloneqq x + 1$

$True \quad True \quad True \quad x \; becomes \; x + 1 \; = 2$

c) **if** $(1 + 1 = 2) \; XOR \; (1 + 2 = 3)$ **then** $x \coloneqq x + 1$

$True \quad False \quad True \quad x \; doesn't \; change$

5. Each inhabitant of a remote village always tells the truth or always lies. A villager will only give "Yes" or a "No" response to a question a tourist asks. Suppose you are a tourist visiting this area and come to a fork in the road. One branch leads to the ruins you want to visit, the other branch leads deeper into the jungle. A villager is standing at the fork in the road. What one question can you ask the villager to determine which branch to take?

   "If I asked 10 minutes ago, is this branch the road to the ruins?"

   - Truth teller will tell "Yes" if the branch leads to the ruins, and "No" if not.
   - If the branch leads to the ruins, liar would tell "No" 10 minutes ago. So, liar will tell "Yes" when the branch leads to the ruins. If the branch leads to the jungle, liar would tell "Yes" 10 minutes ago. So, liar will tell "No" when the branch leads to the jungle.

   ∴ If the villager said "Yes", you could determine this branch to take.

6. Four friends have been identified as suspects for an unauthorized access into a computer system. They have made statements to the investigating authorities. Alice said “Carlos did it.” John said “I did not do it.” Carlos said “Diana did it.” Diana said “Carlos lied when he said that I did it.”

   a) If the authorities also know that exactly one of the four suspects is telling the truth, who did it? Explain your reasoning.

      If Alice is telling the truth, then John is also telling the truth. Contradiction.
      If John is telling the truth, then Diana is also telling the truth. Contradiction.
      If Carlos is telling the truth, then John is also telling the truth. Contradiction.

      If Diana is telling the truth, then John is the criminal.

6. Four friends have been identified as suspects for an unauthorized access into a computer system. They have made statements to the investigating authorities. Alice said “Carlos did it.” John said “I did not do it.” Carlos said “Diana did it.” Diana said “Carlos lied when he said that I did it.”

   b) If the authorities also know that exactly one is lying, who did it? Explain your reasoning.

   If Alice is lying, then Diana is also lying. Contradiction.
   If John is lying, then Carlos is also lying. Contradiction.
   If Diana is lying, then Alice is also lying. Contradiction.

   If Carlos is lying, then Carlos is the criminal.

7. Show that $(p \to q) \wedge (p \to r)$ and $(p \to (q \wedge r))$ are logically equivalent.

$(p \to q) \wedge (p \to r)$

$\equiv (\neg p \vee q) \wedge (\neg p \vee r)$ by truth table for $\to$

$\equiv \neg p \vee (q \wedge r)$ by distributive laws

$\equiv p \to (q \wedge r)$ by truth table for $\to$

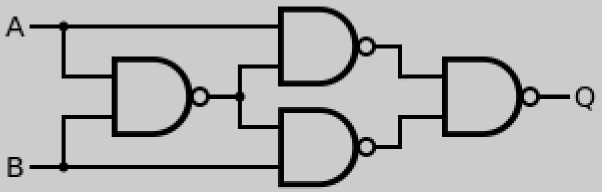
8. Show that $(p \lor q) \land (\neg p \lor r) \rightarrow (q \lor r)$ is a tautology.

$(p \lor q) \land (\neg p \lor r) \longrightarrow (q \lor r)$	
$\equiv \neg((p \lor q) \land (\neg p \lor r)) \lor (q \lor r)$	by truth table for →
$\equiv \neg(p \lor q) \lor \neg(\neg p \lor r) \lor (q \lor r)$	by De Morgan's laws
$\equiv (\neg p \land \neg q) \lor (p \land \neg r) \lor (q \lor r)$	by De Morgan's laws
$\equiv \big((\neg p \land \neg q) \lor q\big) \lor \big((p \land \neg r) \lor r\big)$	by commutative/associative laws
$\equiv \big((\neg p \lor q) \land (\neg q \lor q)\big) \lor \big((p \lor r) \land (\neg r \lor r)\big)$	by distributive laws
$\equiv \big((\neg p \lor q) \land T\big) \lor \big((p \lor r) \land T\big)$	by negation laws
$\equiv (\neg p \lor q) \lor (p \lor r)$	by identity laws
$\equiv (\neg p \lor p) \lor q \lor r$	by commutative/associative laws
$\equiv T \lor q \lor r$	by negation laws
$\equiv T$	by domination laws

8. Show that $(p \lor q) \land (\neg p \lor r) \to (q \lor r)$ is a tautology.

$\boldsymbol{p}$	$\boldsymbol{q}$	$\boldsymbol{r}$	$\boldsymbol{p \lor q}$	$\boldsymbol{\neg p \lor r}$	$\boldsymbol{(p \lor q) \land (\neg p \lor r)}$	$\boldsymbol{(q \lor r)}$
T	T	T	T	T	T	T
T	T	F	T	F	F	T
T	F	T	T	T	T	T
T	F	F	T	F	F	F
F	T	T	T	T	T	T
F	T	F	T	T	T	T
F	F	T	F	T	F	T
F	F	F	F	T	F	F

9. Show that $\{|\}$ (a set of logical operator which contains only $NAND$) is a complete collection of logical operators.

logical operators	***equivalent propositions***
$\neg p$	$p\|p$
$p \wedge q$	$(p\|q)\|(p\|q)$
$p \vee q$	$(p\|p)\|(q\|q)$
$p \rightarrow q$	$\neg p \vee q \equiv \big((p\|p)\|(p\|p)\big)\|(q\|q) \equiv p\ \|(q\|q)$
$p \leftrightarrow q$	$(p \rightarrow q) \wedge (q \rightarrow p) \equiv (p\|(p\|q))\|(q\|(p\|q))$ 

10. Show that $p \mid (q \mid r)$ and $(p \mid q) \mid r$ not equivalent, so that the logical operator $\mid$ is not associative.

$\boldsymbol{p}$	$\boldsymbol{q}$	$\boldsymbol{r}$	$\boldsymbol{q \mid r}$	$\boldsymbol{p \mid q}$	$\boldsymbol{p \mid (q \mid r)}$	$\boldsymbol{(p \mid q) \mid r}$
T	T	T	F	F	T	T
T	T	F	T	F	F	T
T	F	T	T	T	F	F
T	F	F	T	T	F	T
F	T	T	F	T	T	F
F	T	F	T	T	T	T
F	F	T	T	T	T	F
F	F	F	T	T	T	T

11. Let $P(x)$ be the statement "$x + 1 > 2x$." If the domain consists of all integers, what are these truth values?

a) $P(1)$ $P(1): 1 + 1 > 2$ $(False)$

b) $\exists x P(x)$ $P(0): 0 + 1 > 0$ $(True)$

c) $\forall x P(x)$ $x = 1$ doesn't satisfy the condition of $P(x)$ $(False)$

d) $\exists x \neg P(x)$ $x = 1$ satisfy the condition of $\neg P(x)$ $(True)$

12. Express the negation of these propositions using quantifiers, and then express the negation in English.

a) Some driviers do not obey the speed limit.

Let U be all drivers and $P(x)$ be "x obeys the speed limit."

- given proposition : $\exists x \neg P(x)$
- negation of the proposition : $\neg \exists x \neg P(x) \equiv \forall x \neg\neg P(x) \equiv \forall x P(x)$
- English expression : Every driver obeys the speed limit.

b) There is someone in this class who does not have a good attitude.

Let U be all people in this class and $Q(x)$ be "x has a good attitude."

- given proposition : $\exists x \neg Q(x)$
- negation of the proposition : $\neg \exists x \neg Q(x) \equiv \forall x \neg\neg Q(x) \equiv \forall x Q(x)$
- English expression : Everyone in this class has a good attitude.

13. Find a common domain for the variable x, y, z and w for which the statement $\forall x \forall y \forall z \exists w((w \neq x) \land (w \neq y) \land (w \neq z))$ is true and another common domain for these variables for which it is false.

common domain which makes the statement **true** :

- Domains which have at least 4 members.
- e.g.) $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, $\{a, b, c, d\}$

common domain which makes the statement **false** :

- Domains which have at most 3 members.
- e.g.) $\{1, 2\}$, $\{a, b, c\}$

14. Translate these specifications into English where $F(p)$ is "Printer p is out of service," $B(p)$ is "Printer p is busy," $L(j)$ is "Print job j is lost," and $Q(j)$ is "Print job j is queued."

a) $\exists p\big(F(p) \land B(p)\big) \rightarrow \exists jL(j)$

If some printer is out of service and busy, then some print job will be lost.

b) $\forall pB(p) \rightarrow \exists jQ(j)$

If every printer is busy, then some print job will be queued.

c) $\exists j\big(Q(j) \land L(j)\big) \rightarrow \exists pF(p)$

If some print job is queued and lost, then some printer will be out of service.

d) $(\forall pB(p) \land \forall jQ(j)) \rightarrow \exists jL(j)$

If every printer is busy and every print job is queued, then some print job will be lost.

15. Use quantifiers and logical connectives to express the fact that a quadratic polynomial with real number coefficients has at most two real roots.

$a, b, c, x, y, z \in \mathbb{R}$

$$\forall a \forall b \forall c\, \{a \neq 0 \rightarrow \left\{ \forall x \forall y \forall z \begin{pmatrix} (ax^2 + bx + c = 0) \wedge \\ (ay^2 + by + c = 0) \wedge \\ (az^2 + bz + c = 0) \end{pmatrix} \rightarrow \begin{pmatrix} (x = y) \vee \\ (y = z) \vee \\ (z = x) \end{pmatrix} \right\} \}$$

We want to say that for each triple of coefficients (the a, b, and c in the expression $ax^2 + bx + \mathrm{c}(a \neq 0)$, there are at most two values of x making that expression equal to 0.

16. Identify the error or errors in this argument that supposedly shows that if $\forall x(P(x) \lor Q(x))$ is true then $\forall x P(x) \lor \forall x Q(x)$ is true.

1.	$\forall x(P(x) \lor Q(x))$	Premise
2.	$P(c) \lor Q(c)$	Universal instantiation from (1)
3.	$P(c)$	Simplification from (2)
4.	$\forall x(P(x))$	Universal generalization from (3)
5.	$Q(c)$	Simplification from (2)
6.	$\forall x(Q(x))$	Universal generalization from (5)
7.	$\forall x P(x) \lor \forall x Q(x)$	Conjunction from (4) and (6)

17. Use rules of inference to show that if $\forall x(P(x) \rightarrow (Q(x) \wedge S(x)))$ and $\forall x(P(x) \wedge R(x))$ are true, then $\forall x(R(x) \wedge S(x))$ is true.

1. $\forall x(P(x) \rightarrow (Q(x) \wedge S(x)))$	premise
2. $\forall x(P(x) \wedge R(x))$	premise
3. $P(c) \rightarrow (Q(c) \wedge S(c))$	universal instantiation from (1)
4. $P(c) \wedge R(c)$	universal instantiation from (2)
5. $P(c)$	simplification from (4)
6. $Q(c) \wedge S(c)$	modus ponens from (3), (5)
7. $R(c)$	simplification from (4)
8. $S(c)$	simplification from (6)
9. $R(c) \wedge S(c))$	conjunction from (7), (8)
10. $\forall x(R(x) \wedge S(x))$	universal generalization from (9)

18. Show that if n is an integer and $n^3 + 5$ is odd, then n is even using

a) a proof by contraposition.

Assume n is odd and let n be $2k + 1,\ k \in \mathbb{Z}$.
Then $n^3 + 5 = (2k + 1)^3 + 5 = 2(4k^3 + 6k^2 + 3k + 3)$

So, if n is odd , then $n^3 + 5$ is even.
∴ if $n^3 + 5$ is odd, n is even.

b) a proof by contradiction.

Suppose that n is odd. Then n can be represented by $2k + 1,\ k \in \mathbb{Z}$.
Then $n^3 + 5 = (2k + 1)^3 + 5 = 2(4k^3 + 6k^2 + 3k + 3)$

So, $n^3 + 5$ is even.
This contradicts the assumption $n^3 + 5$ is odd.
∴ n is even, when $n^3 + 5$ is odd.

19. Prove that $\sqrt[3]{2}$ is irrational.

Suppose $\sqrt[3]{2}$ is rational. Then there exists integers m and n with $\sqrt[3]{2} = m/n$, where n $\neq 0$ and m and n have no common factors. Then

$$2 = \frac{m^3}{n^3} \qquad 2n^3 = m^3$$

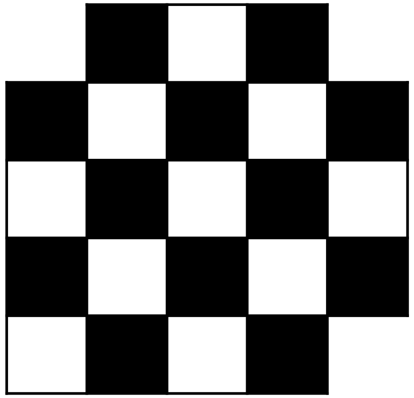
Therefore m^3 must be even. If m^3 is even then m must be even. Since m is even, $m = 2c$ for some integer c. Thus,

$$2n^3 = 8c^3 \qquad n^3 = 4c^3$$

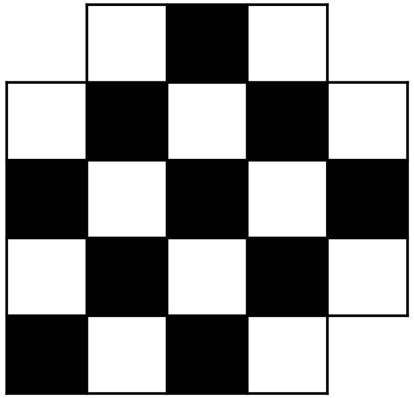
Therefore n^3 is even. Again then n must be even as well. But then 2 must divide both m and n. This contradicts our assumption that m and n have no common factors. We have proved by contradiction that our initial assumption must be false and therefore $\sqrt[3]{2}$ is irrational.

20. Prove or disprove that you can use dominoes to tile a 5×5 checkerboard with three corners removed.

One domino $=$ □■



Suppose we can use dominoes to tile this checkerboard. There are 22 squares in this board and we need 11 dominoes to tile it. We know that each domino cover one black and one white square. So we can cover 11 black and 11 white squares.



But, there are 12 black and 10 white squares or 10 black and 12 white squares.

So it contradicts condition.