Fall 2011 Graduate Course on
Multisensory Predictive Learning

Kalman Filters

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References

- Slides of Michael Williams: see below
- Slides of Chris Williams: see another slide file

Introduction to Kalman Filters

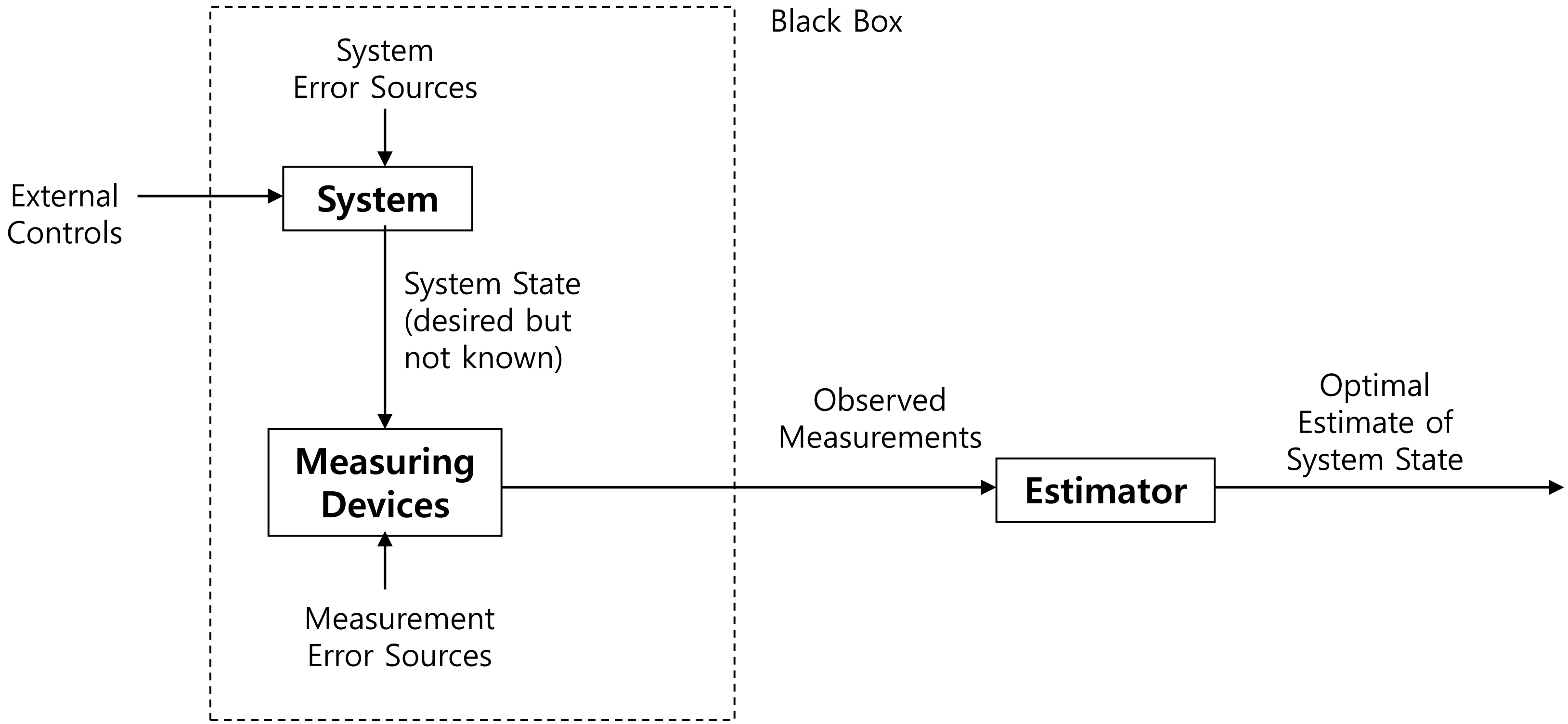
Michael Williams

5 June 2003

Overview

- The Problem – Why do we need Kalman Filters?
- What is a Kalman Filter?
- Conceptual Overview
- The Theory of Kalman Filter
- Simple Example

The Problem



- System state cannot be measured directly
- Need to estimate "optimally" from measurements

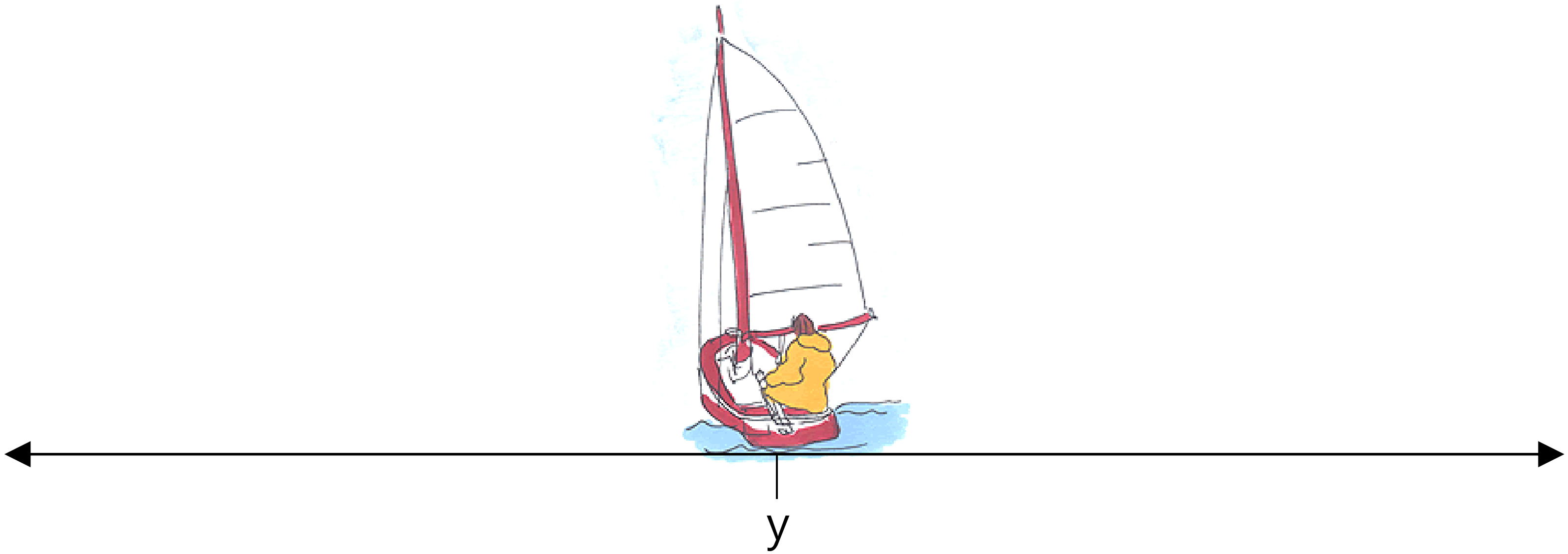
What is a Kalman Filter?

- Recursive data processing algorithm
- Generates optimal estimate of desired quantities given the set of measurements
- Optimal?
  - For linear system and white Gaussian errors, Kalman filter is "best" estimate based on all previous measurements
  - For non-linear system optimality is 'qualified'
- Recursive?
  - Doesn't need to store all previous measurements and reprocess all data each time step

Conceptual Overview

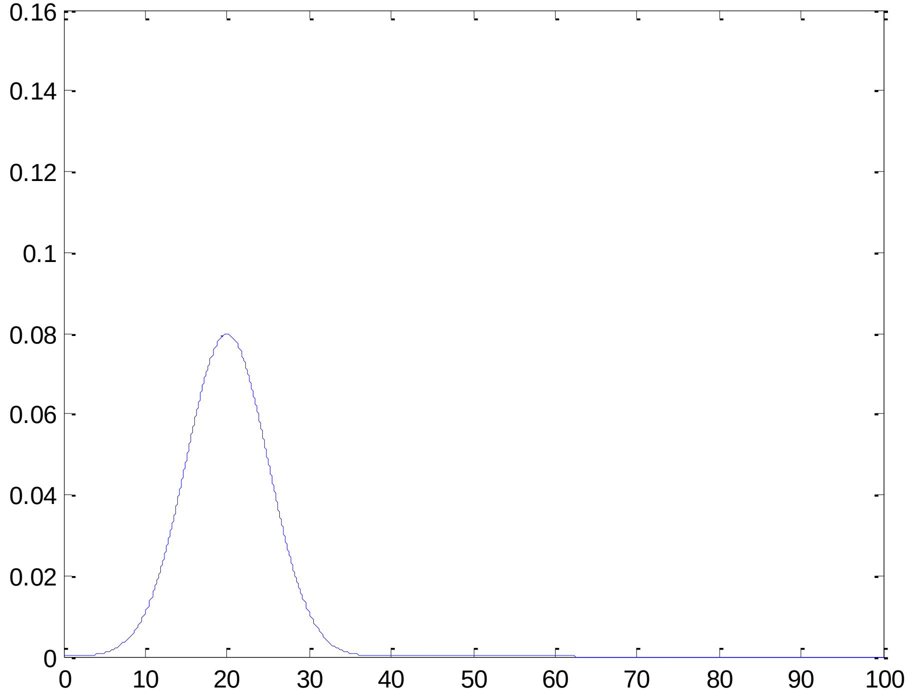
- Simple example to motivate the workings of the Kalman Filter
- Theoretical Justification to come later – for now just focus on the concept
- Important: Prediction and Correction

Conceptual Overview



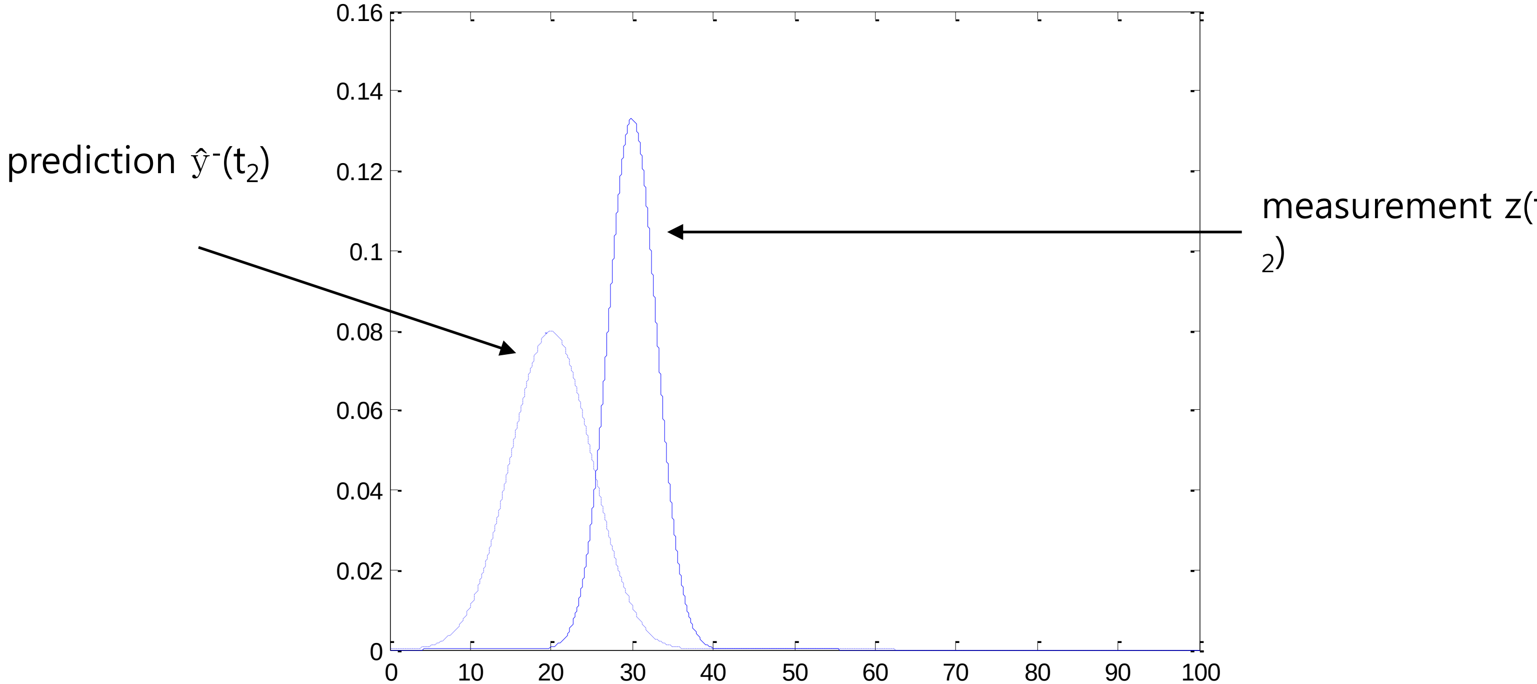
- Lost on the 1-dimensional line
- Position – y(t)
- Assume Gaussian distributed measurements

Conceptual Overview



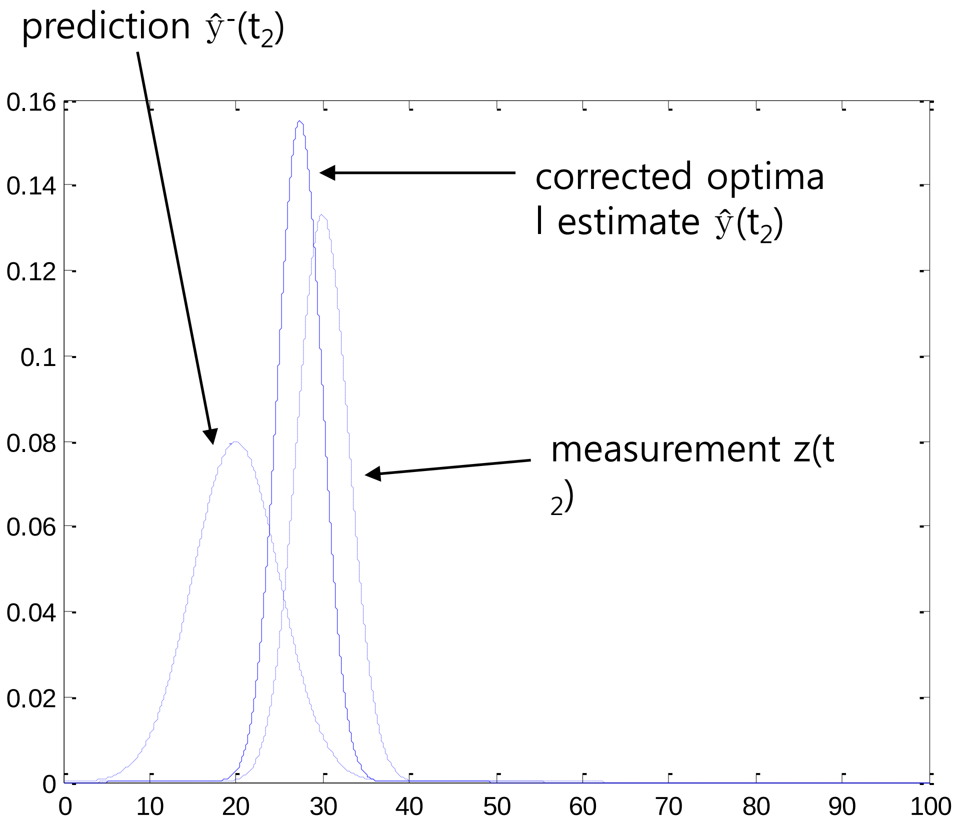
- Sextant Measurement at t_1: Mean = z_1 and Variance = σ_{z1}
- Optimal estimate of position is: $\hat{y}(t_1) = z_1$
- Variance of error in estimate: $\sigma^2_x (t_1) = \sigma^2_{z1}$
- Boat in same position at time t_2 - <u>Predicted</u> position is z_1

Conceptual Overview



- So we have the prediction $\hat{y}^{-}(t_2)$
- GPS Measurement at t_2: Mean = z_2 and Variance = σ_{z2}
- Need to <u>correct</u> the prediction due to measurement to get $\hat{y}(t_2)$
- Closer to more trusted measurement – linear interpolation?

Conceptual Overview



- Corrected mean is the new optimal estimate of position
- New variance is smaller than either of the previous two variances

Conceptual Overview

- Lessons so far:

Make prediction based on previous data - $\hat{y}^-$, σ^-

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Take measurement – z_k, σ_z

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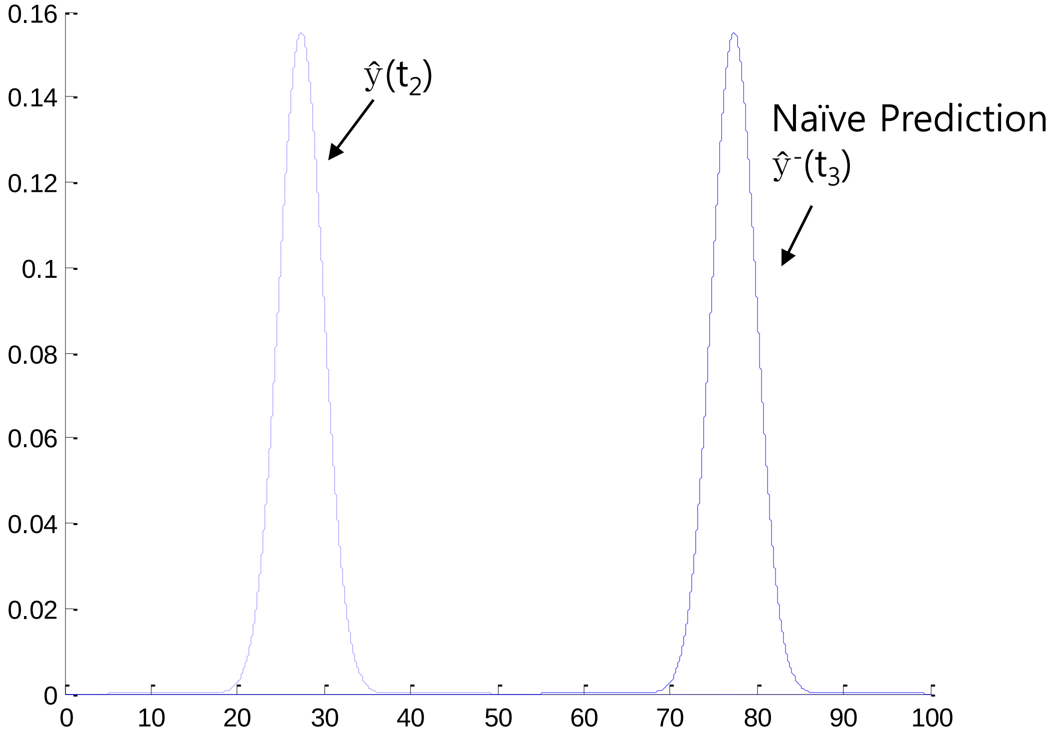
Optimal estimate ($\hat{y}$) = Prediction + (Kalman Gain) * (Measurement - Prediction)

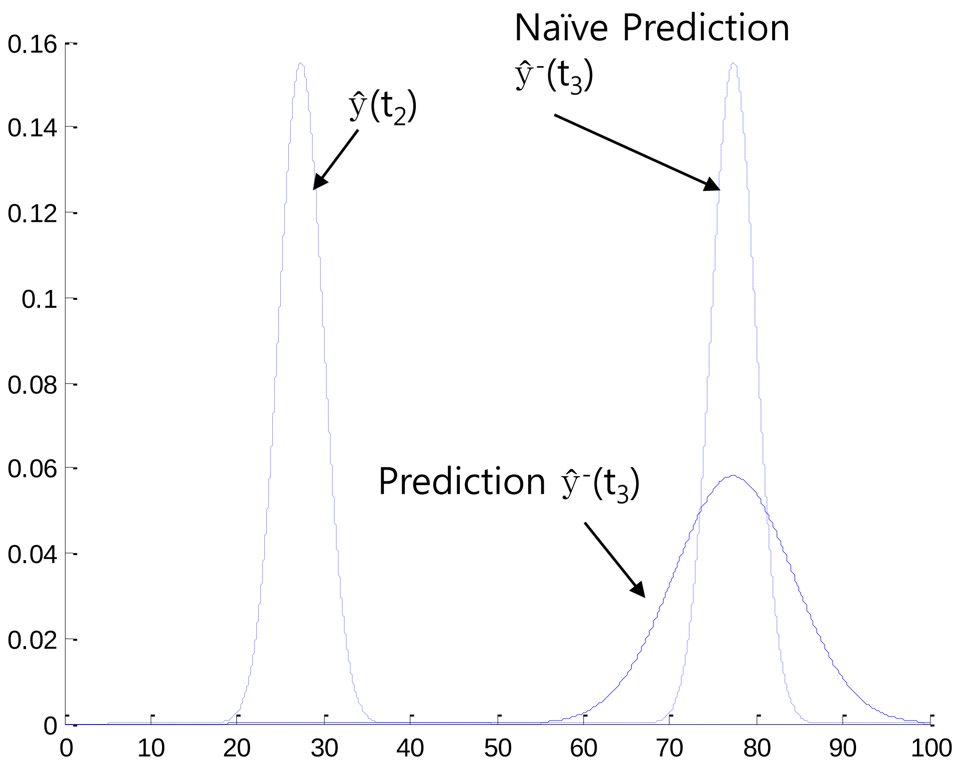
Variance of estimate = Variance of prediction * (1 – Kalman Gain)

Conceptual Overview



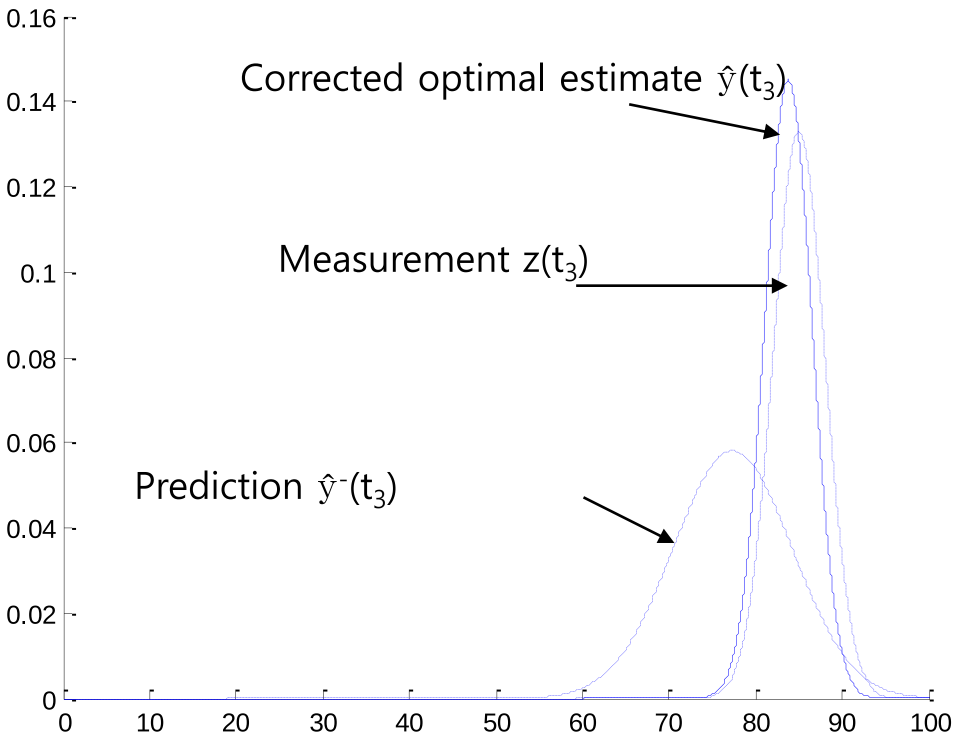
- At time t_3, boat moves with velocity dy/dt=u
- Naïve approach: Shift probability to the right to predict
- This would work if we knew the velocity exactly (perfect model)

Conceptual Overview



- Better to assume imperfect model by adding Gaussian noise
- $dy/dt = u + w$
- Distribution for prediction moves and spreads out

Conceptual Overview



- Now we take a measurement at t_3
- Need to once again correct the prediction
- Same as before

Conceptual Overview

- Lessons learnt from conceptual overview:
  - Initial conditions ($\hat{y}_{k-1}$ and σ_{k-1})
  - Prediction ($\hat{y}^{-}_{k}$, σ^{-}_{k})
    - Use initial conditions and model (eg. constant velocity) to make prediction
  - Measurement (z_k)
    - Take measurement
  - Correction ($\hat{y}_k$, σ_k)
    - Use measurement to correct prediction by 'blending' prediction and residual – always a case of merging only two Gaussians
    - Optimal estimate with smaller variance

Theoretical Basis

- Process to be estimated:

$y_k = Ay_{k-1} + Bu_k + w_{k-1}$ Process Noise (w) with covariance Q

$z_k = Hy_k + v_k$ Measurement Noise (v) with covariance R

- Kalman Filter

Predicted: $\hat{y}^-_k$ is estimate based on measurements at previous time-steps

$$\hat{y}^-_k = Ay_{k-1} + Bu_k$$

$$P^-_k = AP_{k-1}A^T + Q$$

Corrected: $\hat{y}_k$ has additional information – the measurement at time k

$$\hat{y}_k = \hat{y}^-_k + K(z_k - H\hat{y}^-_k)$$

$$K = P^-_k H^T (HP^-_k H^T + R)^{-1}$$

$$P_k = (I - KH)P^-_k$$

Blending Factor

- If we are sure about measurements:
  - Measurement error covariance (R) decreases to zero
  - K decreases and weights residual more heavily than prediction
- If we are sure about prediction
  - Prediction error covariance P^{-}_{k} decreases to zero
  - K increases and weights prediction more heavily than residual

Theoretical Basis

Prediction (Time Update)

(1) Project the state ahead

$$\hat{y}^-_k = Ay_{k-1} + Bu_k$$

(2) Project the error covariance ahead

$$P^-_k = AP_{k-1}A^T + Q$$

Correction (Measurement Update)

(1) Compute the Kalman Gain

$$K = P^-_k H^T (HP^-_k H^T + R)^{-1}$$

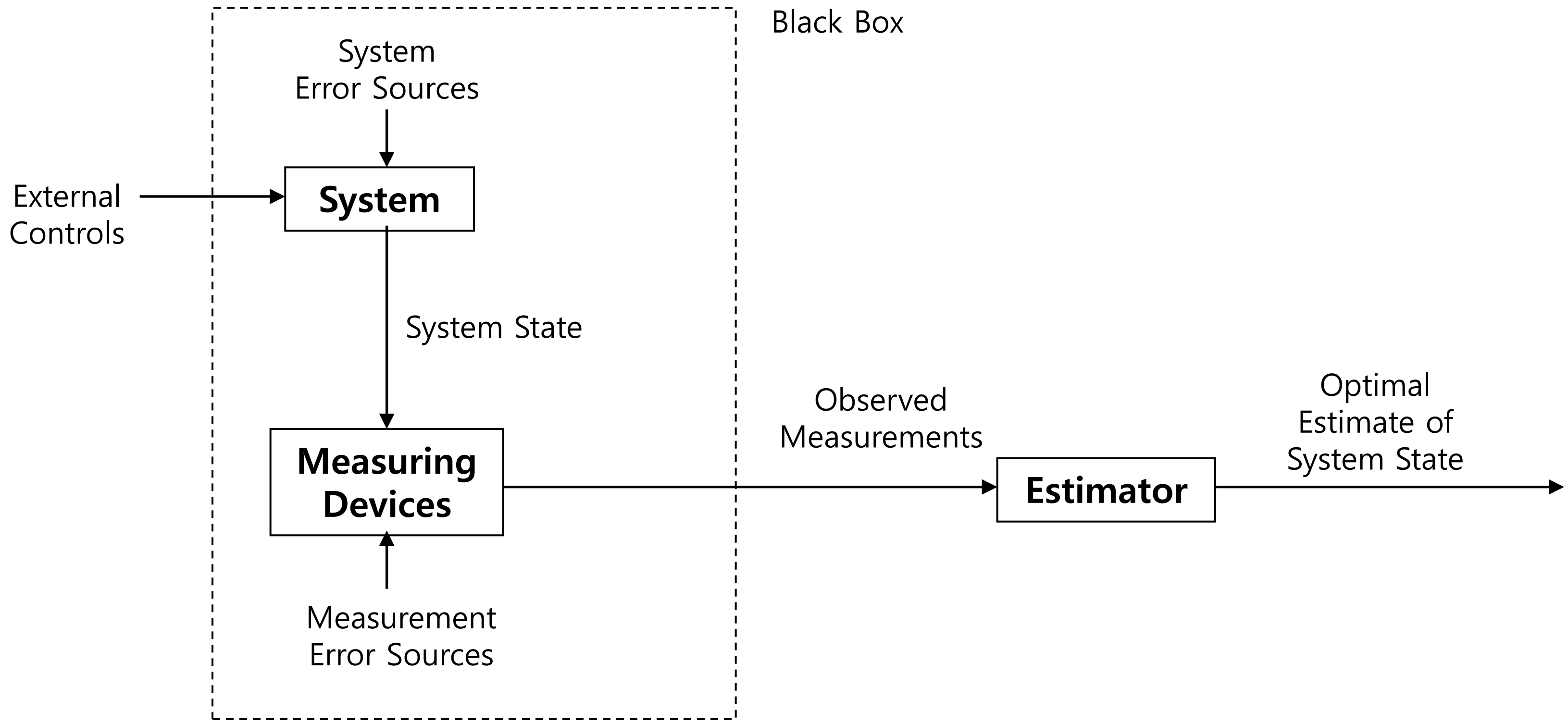
(2) Update estimate with measurement z_k

$$\hat{y}_k = \hat{y}^-_k + K(z_k - H\,\hat{y}^-_k)$$

(3) Update Error Covariance

$$P_k = (I - KH)P^-_k$$

Quick Example – Constant Model



Quick Example – Constant Model

Prediction

$$\hat{y}^-_k = y_{k-1}$$

$$P^-_k = P_{k-1}$$

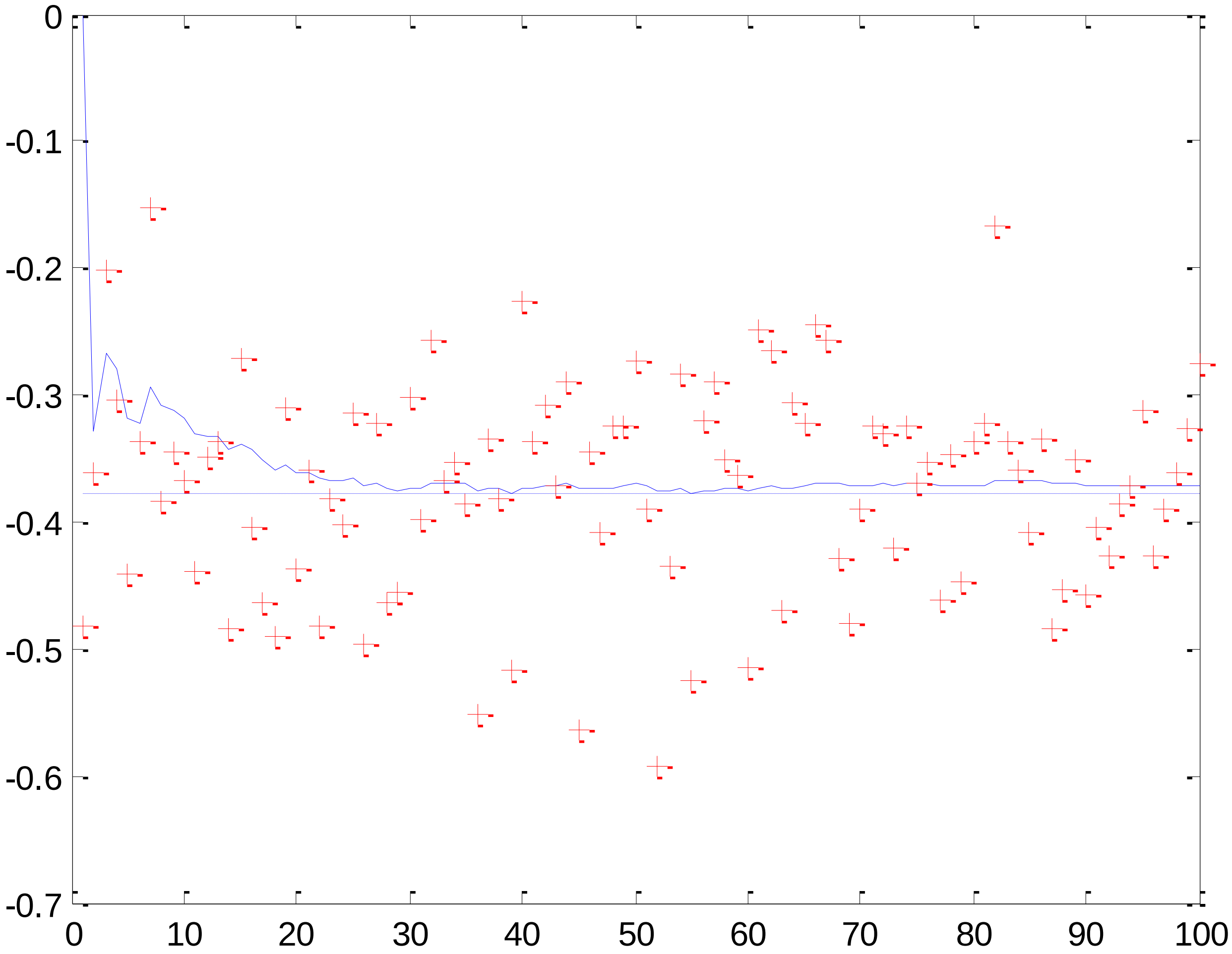
Correction

$$K = P^-_k(P^-_k + R)^{-1}$$

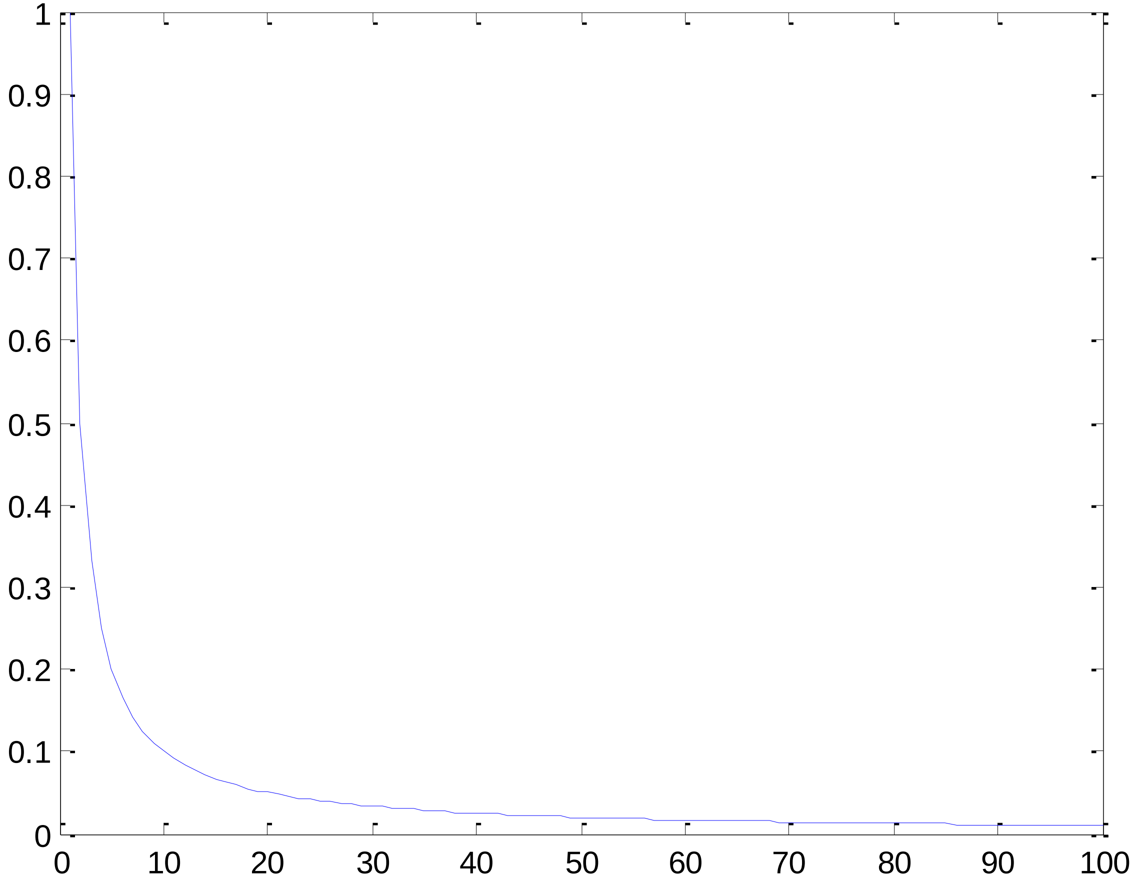
$$\hat{y}_k = \hat{y}^-_k + K(z_k - H\,\hat{y}^-_k)$$

$$P_k = (I - K)P^-_k$$

Quick Example – Constant Model

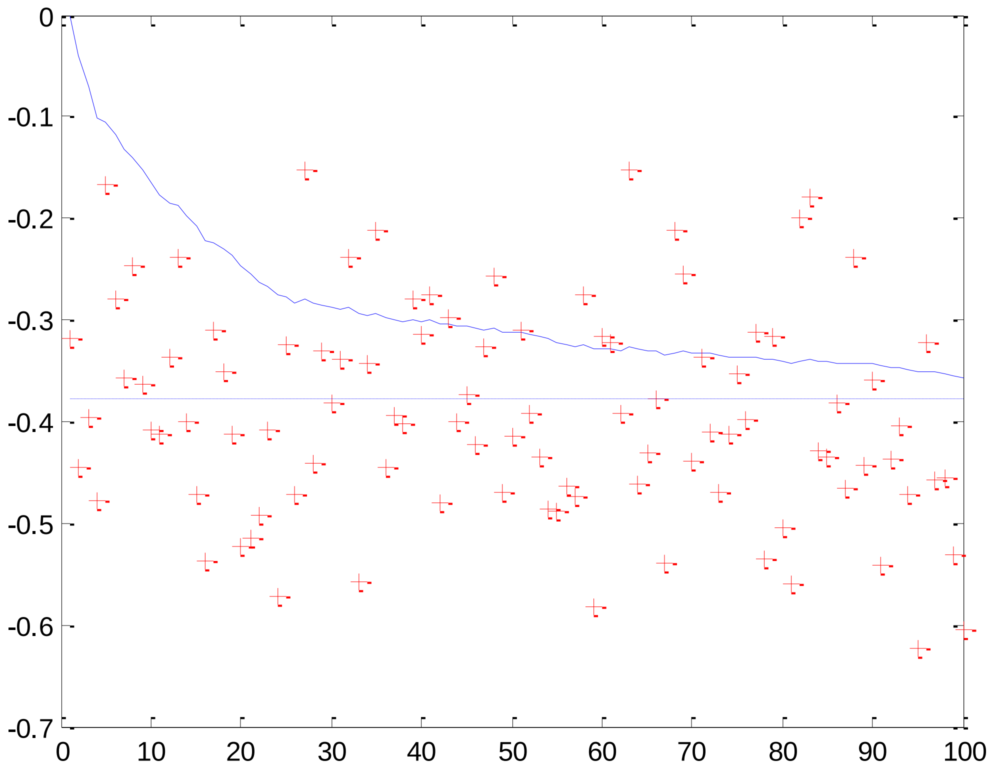


Quick Example – Constant Model



Convergence of Error Covariance - P_k

Quick Example – Constant Model



Larger value of R – the measurement error covariance (indicates poorer quality of measurements)

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Filter slower to 'believe' measurements – slower convergence

References

1. Kalman, R. E. 1960. "A New Approach to Linear Filtering and Prediction Problems", Transaction of the ASME--Journal of Basic Engineering, pp. 35-45 (March 1960).
2. Maybeck, P. S. 1979. "Stochastic Models, Estimation, and Control, Volume 1", Academic Press, Inc.
3. Welch, G and Bishop, G. 2001. "An introduction to the Kalman Filter", http://www.cs.unc.edu/~welch/kalman/