The Dynamical Systems Model of the Simple Genetic Algorithm

7. Crossover and the Mixing-Matrix

- The *effects of crossover* in terms of an operator C acting upon Λ.
  - Given a population $p \in \Lambda$, $(Cp)_k$ will give the probability of individual z_k being in the next population.
  - $G = C \circ U \circ F$
- For all the usual kinds of crossover, the order of crossover and mutation doesn't matter.
  - combination of mutation and crossover gives *mixing scheme* for the GA
  - $M = C \circ U = U \circ C$.
  - $G = M \circ F$
- Some *properties of crossover operator C*
  - $\omega(p,q) \in \Lambda$: population distribution resulting from crossing over random parents from distributions *p* and *q*.
  - $\omega(p,q)_k = \sum_{i,j} p_i q_j \omega(e_i, e_j)_k$: probability that crossing z_i and z_j will produce z_k.
  - ω is an bilinear operator
  - $C(p) = \omega(p,p) = \sum_{i,j} p_i p_j \omega(e_i, e_j)$
  - $C(\alpha p) = \omega(\alpha p, \alpha p) = \alpha\omega(p, \alpha p) = \alpha^2 \omega(p,p) = \alpha^2 C(p)$
  - $G(p) = (M \circ F)(p) = (C \circ U \circ F)(p)$

    $= C(\frac{1}{\bar{f}(p)} USp) = \frac{1}{(\bar{f}(p))^2} C(USp) = \frac{1}{(\bar{f}(p))^2} M(Sp)$
  - C can be defined using *s* matrices, one for each individual z_k.

    $$C(p)_k = \sum_i p_i \sum_j \omega(e_i, e_j)_k p_j = p \cdot (C_k p)$$
  - Utilizing the symmetry of crossover action, C can be written in terms of the matrix C_0 and a set of permutation matrices σ_k

    $$C(p)_k = (\sigma_k p) \cdot C_0 (\sigma_k p) = \sigma_k^T C_0 \sigma_k \quad \text{where} \quad \sigma_k = \begin{cases} 1 \text{ if } z_i \oplus z_k = z_j \\ 0 \text{ otherwize} \end{cases}$$
- The *mixing scheme*

  $$\begin{aligned} M(p)_k &= C(Up)_k = (Up) \cdot (C_k Up) = p \cdot (UC_k Up) \\ &= p \cdot (U\sigma_k^T C_0 \sigma_k Up) = (\sigma_k p) \cdot (UC_0 U\sigma_k p) = (\sigma_k p) \cdot (M_0 \sigma_k p) \end{aligned}$$

  where $M_k = UC_k U$.

8. Mixing and the Walsh Transform

- The *Walsh transform*

  · $W_{i,j} = \frac{1}{\sqrt{s}}(-1)^{z_i \cdot z_j}$ for $i, j = 0, \cdots, (s-1)$.

  · gives us an efficient way to calculate the effects of the operator M.

  · to calculate $G(p)$, we need to calculate the Walsh transform of $M(p)$, which can be calculated easily using Walsh transform of $F(p)$

  · if initial population is chosen uniformly at random from the search space, Walsh transform of $F(p)$ is consisted of fitnesses of each individual.

9. Properties and Conjectures Concerning *G*

- Principle conjecture

  · *G is focused* under assumptions about crossover and mutation.

  · Given any population vector, the sequence $p, Gp, G^2 p, \cdots$ converges

  · Assumptions:

    * Bitwise mutation with a mutation rate < 0.5 and no crossover.

    * Linear fitness function and small mutation rate, when there is crossover.

- Second conjecture

  · Fixed-points of *G* will be *hyperbolic* for nearly all fitness functions.

  · A fixed-point is said to be hyperbolic if the differential of *G* at the point has no eigenvalue with modulus equal to 1.

  · True for fixed-length binary strings, proportionate selection, any kind of crossover, and mutation defined bitwise with a positive mutation rate.

- Properties of *G* from above conjectures

  · There are only finitely many fixed-points of *G*.

  · The probability of picking a population *p*, s.t. iterates of *G* applied to *p* converge on an unstable fixed-point, is zero.

  · The infinite population GA converges to a fixed-point in logarithmic time.